Simultaneous Equations

'Simultaneous' is the name given to these two equations or values. We call them simultaneous because we are trying to find the set of values which satisfy each equation at the *same time* (their 'intersection' or 'solution'). That is, where both equations are true. We look for a 'set' of values because there can be multiple intersections.

Often, we graph these equations on the *same set of axes* as well, so they are 'simultaneously' on the same graph.

Simultaneous equations are also referred to as 'systems of equations'.

Often when we are comparing two or more values or equations that change over time, we want to solve for their intersection. Finding this intersection can give us a lot of valuable information about the nature of the equations.

Example 1

When shopping, we often need to decide on the cheapest and most effective option for our specific needs. Consider two mobile phone plans. They differ in monthly price based on data usage.
Plan 1 is $50 plus $2.75 for every gigabyte of data used.
Plan 2 is $80 plus $0.25 for every gigabyte of data used.
Based on how much mobile data we expect to use each month one of the two plans could be cheaper.

We can express the problem in *Example 1* as two equations that relate usage ('d' for data used) to cost ('C' for cost).

Plan 1: $CC = 50 + 2.75d$

Plan 2: $CC = 80 + 0.25d$

We can see that, for 0GB (0 gigabytes of data used), *Plan 1* is $50 and *Plan 2* is $80, so *Plan 1* is cheaper. For 100GB, however, *Plan 1* is $325 and *Plan 2* is $105, so *Plan 2* is cheaper.
Finding the intersection involves equating the two plans. So, we say that:

$50 + 2.75d = 80 + 0.25d$

Rearranging our equation, we see that:

$30 = 2.5d$, so $d = 12$.

In context, this means that the intersection of the two plans (when both plans have the same cost for the same usage) is at 12GB of data used. From this, we can see that *Plan 1* is cheaper if you expect to use *under* 12GB per month. If we expect to use *more than* 12GB per month, then *Plan 2* is cheaper.

Test your knowledge!

1) Using our findings from *Example 1*, consider which plan is cheapest for the following usage amounts:
   a) 10GB
   b) 20GB
   c) 10GB in one month, then 20GB in the next month

(Refer to the end to check your answers)

Example 1 used only one variable, 'data', but more often we are dealing with two or more variables. There are two ways to solve these problems: *algebraically*, by calculation, or *graphically*, by inspection. The option we use is dependent on the context of the problem and the tools available to us.

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Algebraic Solving

To solve systems of equations algebraically, there are two techniques to be aware of: substitution and elimination.

> ***Important***
>
> When multiplying variables by constants, we can often leave out the multiplication symbols ($\times$ or $\cdot$).
>
> For example, $2 \cdot x = 2x$; $4 \times y = 4y$.
>
> *If you want a better understanding of rearranging and simplifying, check out the* ***Rearranging Equations*** *helpsheet.*

Substitution

Substitution involves replacing one or more parts of an equation with another. Usually this involves 'plugging' an entire equation into another.

> ***Example 2***
>
> Let's take two equations, labelled (1) and (2), respectively:
>
> $2x + y = 4$ $\quad(1)$
>
> $x = 1$ $\quad(2)$
>
> When trying to solve for y we can use substitution.

In trying to solve *Example 2*, we can see that (1) says $2x + y = 4$. From (2), we can see that $x = 1$. So, if $x = 1$, then we can substitute each instance of x in (1) with '1'.

So $\quad 2x + y = 4$

becomes $\quad 2 \cdot 1 + y = 4$

so (1) now becomes $\quad 2 + y = 4$

Rearranging, we find that $y = 2$.

To properly close out the problem, we state the point of intersection clearly, even if it is obvious. In our case, the two equations intersect when $x = 1$ and $y = 2$.

Elimination

Just as we can equate two functions, we can add, subtract, multiply, and divide functions.

> ***Example 3***
>
> Again, let's take two equations, labelled (1) and (2), respectively:
>
> $4x + 2y = 16$ $\quad(1)$
>
> $3x + 2y = 13$ $\quad(2)$
>
> We want to find the values of x and y for which the two equations are true.
> That is, what values for x and y make the statements $'4 \; times \; x \; plus \; 2 \; times \; y \; equals \; 16'$ and $'3 \; times \; x \; plus \; 2 \; times \; y \; equals \; 13'$ true simultaneously.

From *Example 3*, we can take our equation (1) and subtract (2) from it. For each term in each equation, we must only add or subtract *like terms.* When we do, we subtract the coefficient for each term in (2) from it's 'matching' term in (1).
We lay out our equations as follows and subtract our like terms:

$$\begin{array}{ccccc} 4x & + & 2y & = & 16 & \quad (1) \\ - & & - & & - & \\ 3x & + & 2y & = & 13 & \quad (2) \\ = & & = & & = & \\ 1x & + & 0y & = & 3 & \end{array}$$

> ***Important***
>
> Like terms are terms whose variables are the same. For example, $4x$ and $7x$ share the variable x, so they are like terms. The same is true for terms such as $3y$ and y, or $2z^2$ and $3z^2$.

We can then see that $x = 3$ and, now that we have solved for one of our variables, we return to our substitution method to find the other variable, y.
We substitute $x = 3$ into (2) to find:

$$3 \cdot 3 + 2y = 13$$

$$9 + 2y = 13$$

Rearranging, we solve to find that $y = 2$.
Now we know that $x = 3$ and $y = 2$, we have to interpret what that means in the context of our problem. In our case, we are trying to solve for the intersection of our two equations (1) and (2). Therefore, the intersection of (1) and (2) occurs when $x = 3$ and $y = 2$. As an ordered pair that we might plot on a Cartesian plane, we can represent that intersection as occurring at the point $(3, 2)$.

Important

Once we have solved for one variable, we can pick any of the original equations for substitution. Picking the 'easiest' one, with the least amount of calculation needed to solve the remaining variables, is usually a good idea.

We solved *Example 3* in just a few steps; however, it is not always so straightforward. We may need to add instead of subtract, we may also need to add or subtract a *multiple of an equation* from another. In that case, we multiply every term in the equation by the same number.
Consider the following equations (3) and (4):

$$4x + y = 16 \qquad (3)$$

$$3x + 2y = 13 \qquad (4)$$

We cannot simply add or subtract one equation from the other in this case, we must work with multiples of the equations. We can multiply (3) by 2 as follows:

$$4x + y = 16 \qquad (3) \qquad (\times 2)$$

Which becomes:

$$8x + 2y = 32 \qquad (3')$$

We call this new equation $(3')$ (said as "3 prime") because it is based on (3) but is not the same as (3). We can now use $(3')$ together with (4) to solve for x and y:

$$8x + 2y = 32 \qquad (3')$$

$$- \quad - \quad -$$

$$3x + 2y = 13 \qquad (4)$$

$$= \quad = \quad =$$

$$5x + 0y = 19$$

We can then see that $x = \frac{19}{5}$ and we can substitute that into (4) to find:

$$3 \cdot \frac{19}{5} + 2y = 13$$

$$\frac{57}{5} + 2y = 13$$

Rearranging, we find that $2y = \frac{8}{5}$, so $y = \frac{4}{5}$. Now we know that $x = \frac{19}{5}$ and $y = \frac{4}{5}$ which is our intersection.

Test your knowledge!

2) Consider if equation (1) in *Example 3* was $4x + 2y = \mathbf{15}$. How does this affect our solutions? Why does our intersection move? What are the new solutions for x and y?

(Refer to the end to check your answers)

Now that we've learnt how to solve systems of equations algebraically, let's look at solving by inspection, using graphing software.

Solving by Inspection

With appropriate graphing software (such as a graphing calculator or online tools such as Desmos and GeoGebra), we can often find the intersection of two equations without needing to do a lot of arithmetic.

Graphing software will often have tools to find intersections, or we might even be able to see the intersection ourselves (both are forms of 'inspection'). Consider the graph of the two functions, (1) and (2):

$$y = (x - 2)^2 + 1 \qquad (1)$$

$$y = x + 1 \qquad (2)$$

Appropriately detailed graphs of these functions allow us to see that they intersect at the coordinates $(2, 1)$ and $(4, 5)$ (when $x = 2$ and $y = 1$; and when $x = 4$ and $y = 5$). If we were using the software ourselves, there would be a tool to find these for us (often labelled 'intersection', like on a TI-CAS calculator, or you might be able to 'click' on each intersection, like in Desmos).

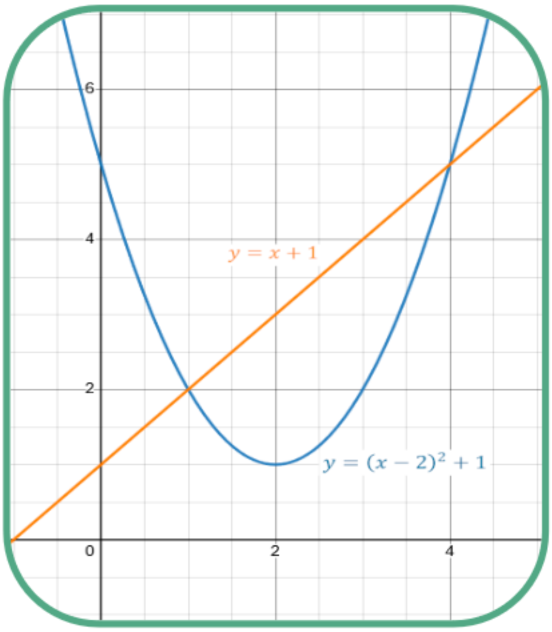

Figure 1 – Graph of (1) *and* (2) *showing intersections at* $(2,1)$ *and* $(4,5)$.

An Important Reminder

Proof by Inspection is often faster than algebraic calculation but can be inaccurate or insufficiently detailed. Our solutions can be presented as approximations rather than exact values ($\frac{4}{3}$ is often approximated as $1.333\ldots$) which can lead to errors. At the same time, graphical representations can help us to build a visual intuition or understanding for the relationship between the equations, but overuse of software can make us reliant on them and reduce our skill in manually solving systems of equations. In certain situations, like paper-based exams, we cannot always use software.

Our solution is to practice both approaches equally and question any result (whether by hand or by machine) with the intent to understand the problem more deeply.

Test your knowledge!

3) We have just solved our system of equations, our graphing software tells us our intersection is at $x = 0.541667$ and $y = 1.346397$. Why would it be a mistake to present these values as our solution? What should we present instead?

(Refer to the end to check your answers)

Answers

Note: The final answer is displayed in **blue**.

1) To find which is cheapest at a particular data amount, we could substitute the usage into each equation and compare, however, we already know the point at which the two plans intersect (12GB of usage) and we know that *Plan 1* is cheaper before the intersection, but *Plan 2* is cheaper after. So, we can just compare if each value is before or after the intersection to find that:

   a) *Plan 1* is cheaper for 10GB of usage.

   b) *Plan 2* is cheaper for 20GB of usage.

   c) There are a few approaches, all of which are valid so long as they arrive at the same answer. One approach is manually totaling the cost for each month for both plans then comparing:

   Plan 1 would cost $182.50 for both months ($77.50 in the first month and $105 in the second) and *Plan 2* would cost $167.50 ($82.50 in the first month and $85 in the second). Thus *Plan 2* is cheaper overall.

   Another approach is to treat it as "one big month" with a total usage of 30GB and a doubled starting fee. Using this approach, we also find that:

   Plan 1 would cost $182.50 for both months ($100 starting fee and $82.50 in data usage) and *Plan 2* would cost $167.50 ($160 starting fee and only $7.50 in data usage). Thus *Plan 2* is cheaper overall.

2) If our initial equations change, their intersection is very unlikely to remain the same. In our case, we can now solve:

$$
\begin{array}{ccccccl}
4x & + & 2y & = & 15 & & (1) \\
- & & - & & - & & \\
3x & + & 2y & = & 13 & & (2) \\
= & & = & & = & & \\
1x & + & 0y & = & 2 & &
\end{array}
$$

   This means $x = 2$. So, substituting into (2), we find:

$$
\begin{aligned}
3 \cdot 2 + 2y &= 13 \\
6 + 2y &= 13
\end{aligned}
$$

   So, $y = \frac{7}{2}$. Therefore, our intersection is now at $x = 2$ and $y = \frac{7}{2}$.

3) We cannot be sure that $x = 0.541667$ and $y = 1.346397$ are exact solutions to our equations. If we present these as the intersection, we may be incorrect, and if we are to perform more calculations based on this result, our error will become larger. Instead, we should try to solve the system of equations manually to fin the *exact solution* to our equations, which are $x = \frac{13}{24}$ and $y = \frac{3\pi}{7}$.

Related helpsheets

- Rearranging Equations helpsheet